K24P 4043

Reg. No. : ....

Name : ....

I Semester M.Sc. Degree (CBSS – Supplementary) Examination, October 2024 (2021 and 2022 Admissions) MATHEMATICS MAT1C04 : Basic Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **four** questions from this part. Each question carries 4 marks.

1. Let (X, d) be a metric space, let $x \in X$ and let $\epsilon > 0$. Prove that $A = \{y \in X : d(x, y) \leq \epsilon\}$ is a closed subset of X.

2. Prove that every second countable space is separable. Is the converse true ? Justify your answer with an example.

3. Let $(A, \mathscr{T}_A)$ be a subspace of a topological space $(X, \mathscr{T})$. Prove that a subset C of A is closed in $(A, \mathscr{T}_A)$ if and only if there is a closed subset D of $(X, \mathscr{T})$ such that $C = A \cap D$.

4. Let $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ be topological spaces, and let $(X_1 \times X_2, \mathscr{T})$ be the product space. Prove that the projection maps are continuous. Also show that the product topology is the smallest topology for which both projections are continuous.

5. A topological space $(X, \mathscr{T})$ is connected if and only if no nonempty proper subset of X is both open and closed.

6. Define Cantor set. **(4×4=16)**

P.T.O.

PART – B

Answer **four** questions from this part **without** omitting **any** Unit. **Each** question carries **16** marks.

Unit – I

7. a) Let $\{\mathscr{T}_\alpha : \alpha \in \Lambda\}$ be a collection of topologies on a set X. Prove that $\cap\{\mathscr{T}_\alpha : \alpha \in \Lambda\}$ is a topology on X.

   b) Let X be a set and let $\mathscr{S}$ be a collection of subsets of X such that $X = \cup\{S : S \in \mathscr{S}\}$. Prove that there is a unique topology $\mathscr{T}$ on X such that $\mathscr{S}$ is a subbasis for $\mathscr{T}$.

   c) Let $X = \{1, 2, 3, 4, 5\}$ and $\mathscr{S} = \{\{1\}, \{1, 2, 3\}, \{2, 3, 4\}, \{3, 5\}\}$. Prove that $\mathscr{S}$ is a subbasis for a topology on X. Also find $\mathscr{T}$.

8. a) Let A and B be subsets of a topological space $(X, \mathscr{T})$. Prove that :

   i) A is open if and only if A = int A.

   ii) int (A) $\subseteq$ int (B) whenever $A \subseteq B$.

   iii) int $(A \cap B)$ = int (A) $\cap$ int(B).

   iv) int (A) $\cup$ int (B) $\subseteq$ int $(A \cup B)$.

   b) Let $n \in \mathbb{N}$ and $\mathscr{T}$ is the usual topology on $\mathbb{R}^n$. Prove that $(\mathbb{R}^n, \mathscr{T})$ is second countable.

9. a) Let $(X, \mathscr{T})$ be a topological space, Let $A \subset X$ and let $x \in X$. Prove that

   i) if there is a sequence of points of A that converges to x, then $x \in \bar{A}$.

   ii) if $(X, \mathscr{T})$ is first countable and $x \in \bar{A}$, then there is a sequence of points of A that converges to x.

   b) Let (X, d) be a complete metric space and let A be a subset of X with subspace metric $\rho = d|_{(A \times A)}$. Prove that (A, ρ) is complete if and only if A is a closed subset of X.

   c) Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces and let $f : X \to Y$. Suppose $(X, \mathscr{T})$ is first countable and for each $x \in X$ and each sequence $\langle x_n \rangle$ such that $\langle x_n \rangle \to x$, the sequence $\langle f(x_n) \rangle \to f(x)$. Then prove that f is continuous.

Unit – II

10. a) Prove that the topological properties Hausdorff and metrizability are hereditary.

b) Let $\{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of first countable spaces and let $X = \prod_{\alpha \in \Lambda} X_\alpha$. Prove that $(X, \mathscr{T})$ is first countable if and only if $\mathscr{T}_\alpha$ is the trivial topology for all but a countable number of α.

11. a) Give an example to show that separability is not hereditary.

b) State and prove Pasting lemma.

c) Let $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ be topological spaces, and for i = 1, 2 let $\mathscr{B}_i$ be bases for $\mathscr{T}_i$. Then prove that $\mathscr{B} = \{U \times V : U \in \mathscr{B}_1 \text{ and } V \in \mathscr{B}_2\}$ is a basis for the product topology $\mathscr{T}$ on $X_1 \times X_2$.

12. a) Let $\{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces, and for each $\alpha \in \Lambda$, let $(A_\alpha, \mathscr{T}_{A\alpha})$ be a subspace of $(X_\alpha, \mathscr{T}_\alpha)$. Then prove that the product topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is the same as the subspace topology on $\prod_{\alpha \in \Lambda} A_\alpha$ is determined by the product topology on $\prod_{\alpha \in \Lambda} X_\alpha$.

b) Let $\{(Y_\alpha, \mathscr{U}_\alpha) : \alpha \in \Lambda\}$ be an indexed family of topological spaces. Let $\mathscr{U}$ be the product topology on $Y = \prod_{\alpha \in \Lambda} Y_\alpha$, let $(X, \mathscr{T})$ be a topological space, and let $f : X \to Y$ be a function. Prove that f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in \Lambda$.

Unit – III

13. a) Let $\mathscr{T}$ be the usual topology on $\mathbb{R}$. Prove that $(\mathbb{R}, \mathscr{T})$ is connected.

b) State and prove intermediate value theorem.

c) Prove that the Cantor set is totally disconnected.

14. a) Prove that the fixed point property is a topological invariant.

b) Prove that the topologist's sine curve is not pathwise connected.

15. a) Let $\{(A_\alpha, \mathscr{T}_{A\alpha}) : \alpha \in \Lambda\}$ be a collection of connected subspaces of a topological space $(X, \mathscr{T})$ and let $A = \cup_{\alpha \in \Lambda} A_\alpha$. Then prove that

i) If $\cap_{\alpha \in \Lambda} A_\alpha \neq \varnothing$ then $(A, \mathscr{T}_A)$ is connected.

ii) If $\Lambda = \mathbb{N}$ and $A_n \cap A_{n+1} \neq \varnothing$ for each $n \in \mathbb{N}$, then $(A, \mathscr{T}_A)$ is connected.

b) Prove that a topological space $(X, \mathscr{T})$ is locally connected if and only if each component of each open set is open.

c) Prove that every 0-dimensional T_0 space is totally disconnected. **(4×16=64)**