K19P 0357

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (Reg./Suppl./Imp.) Examination, April 2019
(2017 Admission Onwards)
MATHEMATICS
MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries 4 marks.

1. Show that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$ for any B.

2. Prove that the set of irrationals in the interval [1, 4] is Lebesgue measurable and has a measure 3.

3. Show that $\int_1^\infty dx / x = \infty$.

4. If f is non negative measurable function, then prove that f = 0 a.e. if and only if $\int f \, dx = 0$.

5. Let $[\![X, S, \mu]\!]$ be a measure space and $E_1, E_2 \in S$. Show that $\mu(E_1 \Delta E_2) = 0$ implies $\mu(E_1) = \mu(E_2)$.

6. Show that if $\mu(X) < \infty$ and $0 < p < q \leq \infty$, then $L^p(\mu) \subseteq L^q(\mu)$. **(4×4=16)**

PART – B

Answer **any four** questions from this Part without omitting any Unit. **Each** question carries **16** marks.

Unit – I

7. a) Prove that the outer measure of an interval is its length.
   b) Prove that outer measure is translation invariant.
   c) For any set A and any $\varepsilon > 0$, show that there is an open set 0 containing A and such that $m^*(0) \leq m^*(A) + \varepsilon$.

P.T.O.

8. a) Prove that there exists a non measurable set.

   b) Let T be a measurable set of positive measure and let $T^* = [x - y : x \in T, y \in T]$. Show that T^* contains an interval $(-\alpha, \alpha)$ for some $\alpha > 0$.

9. a) Let f be a non negative measurable function. Then prove that there exists a sequence $\{\varphi_n\}$ of simple functions such that, for each x, $\varphi_n(x) \uparrow f(x)$.

   b) Let f and g are non negative measurable functions. Then prove that $\int f\,dx + \int g\,dx = \int (f + g)\,dx$.

Unit – II

10. a) State and prove Lebesgue's dominated convergence theorem.

    b) If f is Riemann integrable and bounded over the finite interval [a, b], then prove that f is integrable and $R\int_a^b f\,dx = \int_a^b f\,dx$.

11. a) Show that $f \in L\,(a + h, b + h)$ and $f_h(x) \equiv f(x + h)$, then prove that $f_h \in L\,(a, b)$ and $\int_{a+h}^{b+h} f\,dx = \int_a^b f\,dx$.

    b) Let f be a bounded measurable function defined on the finite interval (a, b). Show that $\lim_{\beta \to \infty} \int_a^b f(x) \sin \beta x\,dx = 0$.

    c) Show that Lebesgue integrable function need not be Riemann integrable.

12. a) Let μ^* be an outer measure on $\mathcal{H}(\mathcal{R})$ and let S^* denote the class of μ^* – measurable sets. Then prove that S^* is a σ-ring and μ^* restricted to S^* is a complete measure.

    b) Prove that μ is σ-finite measure on a ring $\mathcal{R}$, then prove it has a unique extension to the σ-ring $S(\mathcal{R})$.

Unit – III

13. a) Let $[\![X, S, \mu]\!]$ be a measure space and $Y \in S$. Let S_Y consist of those sets of S that are contained in S. Define $\mu_Y(E) = \mu(E)$ if $E \in S_Y$. Then show that $[\![Y, S_Y, \mu_Y]\!]$ is a measure space.

b) Show that $L^p(\mu)$ is a vector space.

14. a) State and prove Minkowski's inequality.

b) If $\rho(f, g) = \| f - g \|_p$ then prove that ρ is a metric on $L^p(\mu)$.

c) Let $p \geq 1$ and let $\| f_n - f \|_p \to 0$. Show that $\| f_n \|_p \to \| f \|_p$.

15. a) Prove that if $\{f_n\}$ is a sequence in $L^\infty(\mu)$ such that $\| f_n - f_m \|_\infty \to 0$ as $n, m \to \infty$, then there exists a function f and such that $\lim f_n = f$ a.e., $f \in L^\infty(\mu)$ and $\| f_n - f \|_\infty \to 0$.

b) Let $[\![X, S, \mu]\!]$ be a measure space and $E_n \in S$, $n = 1, 2, \ldots$ Show that

i) $\mu(\liminf E_n) \leq \liminf \mu(E_n)$.

ii) If $\mu(X) < \infty$ then $\limsup \mu(E_n) \leq \mu(\limsup E_n)$. **(4×16=64)**
